MULTIOBJECTIVE OPTIMIZATION INVOLVING QUADRATIC FUNCTIONS

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Abstract

Multiobjective optimization is nowadays a word of order in engineering projects. Although the idea involved is simple, the implementation of any procedure to solve a general problem is not an easy task. Evolutionary algorithms are widespread as a satisfactory technique to find a candidate set for the solution. Usually they supply a discrete picture of the Pareto front even if this front is continuous. In this paper we propose three methods for solving unconstrained multiobjective optimization problems involving quadratic functions. In the first, for bi-objective optimization defined in the bi-dimensional space, a continuous Pareto set is found analytically. In the second, applicable to multiobjective optimization, a condition test is proposed to check if a point in the decision space is Pareto optimum or not and, in the third, with functions defined in *n*-dimensional space, a direct non-iterative algorithm is proposed to find the Pareto set. Simple problems highlight the suitability of the proposed methods.

Keywords

Multiobjective optimization; Quadratic functions; Continuous Pareto set.

1. Introduction

Life is about making decisions and the choice of the optimal solutions is not an exclusive subject of scientists, engineers and economists. Decision making is present in day-to-day life. Looking for an enjoyable vacancy, everyone will formulate an optimization problem to a travel agent, a problem like: with a minimum amount of money, visit a maximum number of places in a minimum amount of time and with the maximum of comfort. Usually all real design problems have more than one objective, namely they are multiobjective. Moreover, the design objectives are often antagonistic.

Edgeworth (1881) was the pioneer to define an optimum for multi-criteria economic decision making problem, at King's College, London. It was about the multi-utility problem within the context of two consumers, P and π: "*It is required to find a point (x, y) such that in whatever direction we take an infinitely small step, P and π do not increase together but that, while one increases, the other decreases.*"

Few years later, in 1896, Pareto (1971), at the University of Lausanne, Switzerland, formulated his two main theories, Circulation of the Elites and The Pareto Optimum: "*The optimum allocation of the resources of a society is not attained so long as it is possible to make at least one individual better off in his own estimation while keeping others as well off as before in their own estimation.*"

Since then, many researchers have been dedicated to developing methods to solve this kind of problem. Interestingly, solutions for problems with multiple objectives, also called multi-criteria optimization or vector optimization, are treated as Pareto optimal solutions or Pareto front, although as Stadler (1988) observed, they should be treated as Edgeworth-Pareto solutions.

Extensive reviews of multiobjective optimization concepts and methods are given by Miettinen (1998); for evolutionary algorithms by Goldberg (1989) and for evolutionary multiobjective optimization by Deb (2001). The theoretical basis for multiobjective optimization adopted in this work was based on these references.

Thanks to the computer development optimization of large scale problems became a common task in engineering designs. The development of high speed computers and its increasing use in several industrial branches led to significant changes in the design processes. Currently, the computers, each time faster, allow the engineer to consider a wider range of design possibilities and optimization processes allow systematic choice between alternatives, since they are based on some rational criteria. If used adequately, these procedures can, in most cases, improve or even generate the final results of a design.

Associated to computer development, many of the research done in optimization is focused on numerical methods to solve any kind of problem but sometimes simplified problems can give important clues to the designer during the trade-off phases of a decision.

The present work aims to bring new approaches to solve multiobjective optimization problems, providing a rapid solution for the Pareto set if the objective functions involved are quadratic.

Nomenclature

DM	decision maker
$\mathbf{f}(\mathbf{X})$	objective functions vector
GA	genetic algorithm
$g_j(\mathbf{X})$	j^{th} inequality constraint function
$h_i(\mathbf{X})$	i^{th} equality constraint function

k	number of objective functions
KKT	Karush-Khun-Tucker
l	number of equality constraint functions
m	number of inequality constraint functions
MOOP	multiobjective optimization problem
NSGA II	non-dominated sorting genetic algorithm, version two
n	dimension of the design space
$\mathbb{R}^k$	function or criterion space
$\mathbb{R}^n$	decision variables or design space
$\mathbb{S}$	feasible region in the design space
x_i	i^{th} decision variable
$\mathbf{X}$	decision variable vector
$\mathbf{X}^*$	non-dominated solution of a multiobjective optimization problem
$\mathbf{X}_{inf}, \mathbf{X}_{sup}$	lower and upper bounds of the design space
ω_i	weighting factor for the i^{th} objective function gradient in KKT condition
$\boldsymbol{\omega}$	vector of ω_{is}
λ_j	weighting factor for j^{th} inequality constraint gradient in KKT condition
$\boldsymbol{\lambda}$	vector of λ_{js}
μ_i	weighting factor for i^{th} equality constraint gradient in KKT condition
$\boldsymbol{\mu}$	vector of μ_{is}
∇	gradient operator

2. Multiobjective Optimization Problem

Multiobjective optimization problems (MOOP) can be defined by the following equations:

$$\textit{minimize:} \quad \mathbf{f}(\mathbf{X}) \tag{1.a}$$

$$\textit{subject to:} \quad g_i(\mathbf{X}) \leq 0, i = 1,2, \dots m. \tag{1.b}$$

$$h_j(\mathbf{X}) = 0, j = 1,2, \dots l. \tag{1.c}$$

$$\mathbf{X}_{inf} \leq \mathbf{X} \leq \mathbf{X}_{sup} \tag{1.d}$$

where $\mathbf{f}(\mathbf{X}) = [\, f_1, f_2, f_3, \dots f_k\,]^T: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a vector with the values of scalar objective functions $f_i(\mathbf{X}): \mathbb{R}^n \rightarrow \mathbb{R}$ to be minimized. $\mathbf{X} \in \mathbb{R}^n$ is the vector containing the design variables, also called decision variables, defined in design space $\mathbb{R}^n$. $\mathbf{X}_{inf}$ and $\mathbf{X}_{sup}$ are respectively the lower and upper bounds of the design variables. $g_i(\mathbf{X}): \mathbb{R}^n \rightarrow \mathbb{R}$ represents the i^{th} inequality constraint function and $h_j(\mathbf{X}): \mathbb{R}^n \rightarrow \mathbb{R}$ the j^{th} equality constraint function. Equations (1.b) to (1.d), define the region of feasible solutions, $\mathbb{S}$, in design space $\mathbb{R}^n$. The constraints $g_i(\mathbf{X})$ are of type "*less than or equal*" functions in view of the fact that "*greater or equal*" functions may be converted to the first type if they are multiplied by -1. Similarly, the problem considers the "*minimization*" of $f_i(\mathbf{X})$, given that function "*maximization*" can be transformed into the former by multiplying it by -1.

2.1. Pareto optimal solution

The notion of "optimum" in solving problems of multiobjective optimization is known as "Pareto optimal." A solution is said to be Pareto optimal if there is no way to improve one objective without worsening at least one other, i.e., the feasible point $\mathbf{X}^* \in \mathbb{S}$ is Pareto optimal if there is no other feasible point $\mathbf{X} \in \mathbb{S}$ such that $\forall\, i, j$ with $i \neq j$, $f_i(\mathbf{X}) \leq f_i(\mathbf{X}^*)$ with strict inequality in at least one condition, $f_j(\mathbf{X}) < f_j(\mathbf{X}^*)$.

Due to the conflicting nature of the objective functions, the Pareto optimal solutions are usually scattered in the region $\mathbb{S}$, a consequence of not being able to minimize all the objective functions simultaneously. In solving the optimization problem we obtain the Pareto set or the Pareto optimal solutions defined in the design space, and the Pareto front, an image of the objective functions, in the criterion space, calculated over the set of optimal solutions.

2.2. Necessary condition for Pareto optimality

In fact, optimizing multiobjective problems expressed by Eqs. (1.a)-(1.d) is of general character. The equations represent the problem of single-objective optimization when $k = 1$. According to Miettinen (1998), such as in single-objective optimization problems, the solution $\mathbf{X}^* \in \mathbb{S}$ for the Pareto optimality must satisfy the Karush-Kuhn-Tucker condition (KKT), expressed as:

$$\sum_{i=1}^{k} \omega_i \nabla f_i(\mathbf{X}^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{X}^*) + \sum_{i=1}^{l} \mu_i \nabla h_i(\mathbf{X}^*) = 0 \tag{2.a}$$

$$\lambda_j g_j(\mathbf{X}^*) = 0 \tag{2.b}$$

$$\lambda_j \geq 0 \tag{2.c}$$

$$\mu_i \geq 0 \tag{2.d}$$

$$\omega_i \geq 0; \ \sum_{i=1}^{k} \omega_i = 1 \tag{2.e}$$

where ω_i is the weighting factor for the gradient of the i^{th} objective function, calculated at the point $\mathbf{X}^*$, $\nabla f_i(\mathbf{X}^*)$. λ_j represents the weighting factor for the gradient of the j^{th} inequality constraint function, $\nabla g_j(\mathbf{X}^*)$, and is zero when the constraint function associated is not active, i.e., $g_j(\mathbf{X}^*) \leq 0$. μ_i represents the weighting factor for the gradient of the i^{th} equality constraint function, $\nabla h_i(\mathbf{X}^*)$.

Equations (2.a) to (2.e) form the necessary conditions for $\mathbf{X}^*$ to be a Pareto optimal. As described by Miettinen (1998), it is sufficient for the complete mapping of the Pareto front if the problem is convex and the objective functions are continuously differentiable in the $\mathbb{S}$ space. Otherwise, the solution will depend on additional conditions, as shown by Marler and Aurora (2004).

Some researchers have attempted to classify methods for solving MOOP according various considerations. Hwang and Masud (1979) and later Miettinen (1998) suggested the following four classes, depending on how the decision maker (DM) articulates preferences: no-preference methods, a priori methods, a posteriori methods, and interactive methods.

In no-preference articulation methods, the preferences of the DM are not taken into consideration. The problem can be solved by a simple method and the solution obtained is presented to the DM which will accept or reject it.

In a priori preference articulation methods, the hopes and opinions of the DM are taken into consideration before the solution process. Those methods require that the DM knows beforehand the priority of each objective transforming the multi-objective problem in a single-objective problem where the function to be optimized is a combination of objective functions.

In posteriori preference articulation methods no preferences of the DM are considered. After the Pareto set has been generated, the DM chooses a solution from this set of alternatives.

In interactive preference articulation methods the DM preferences are continuously used during the search process and are adjusted as the search continues.

The methods we will propose in the next sections can be classified in posteriori preference articulation and an extensive literature review of the most important methods to solve multiobjective optimization problems can be found in Augusto et al (2012).

3. Two dimensional functions of Class C[1]

In this section we propose a simple strategy to determine de Pareto set in the decision space and the corresponding Pareto front in the function space, for MOOP involving two bi-dimensional differentiable functions.

Consider an unconstrained multiobjective optimization problem. From Eq. (2.a), the optimality condition can be interpreted by the following proposition:

> ***Proposition 1***. *If exists a Pareto front for the minimization problem with two continuous and differentiable functions defined in $\mathbb{R}^2$, say $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$, then the curve connecting the minima of both functions and is orthogonal to the function contours in the decision space defines the Pareto set.*

As the gradients of each function are orthogonal to contours and point outwards from the minimum, the curve mentioned in *Proposition 1* is the locus where the gradients of both functions are parallel and opposite, as shown in Fig. 1.

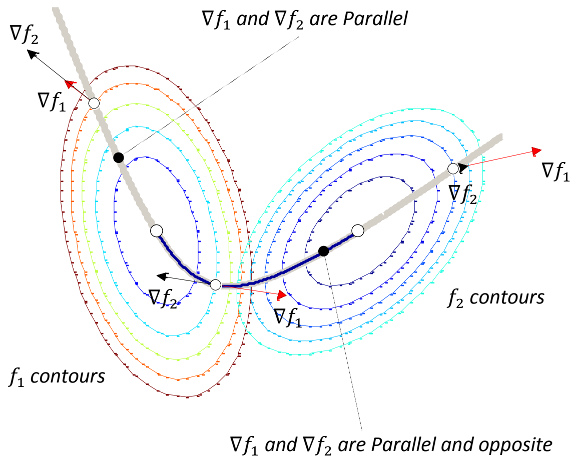


Figure 1 - Graphical representation of *Proposition 1*. The continuous Pareto set as the locus where objective function gradients are parallel and opposite

3.1. Two quadratic functions defined in $\mathbb{R}^2$ space

Proposition 1 is quite general but as our focus is on quadratic functions let us solve an unconstrained bi-objective optimization problem involving quadratic functions defined in the two dimensional decision space, i.e., $\mathbf{f}(x_1, x_2) = [f_1, f_2]: \mathbb{R}^2 \to \mathbb{R}^2$. The problem is defined as:

minimize:

$$f_1(x_1, x_2) = a_1 x_1^2 + (b_1 x_1 + e_1)x_2 + c_1 x_2^2 + d_1 x_1 + cst_1 \quad (3.a)$$
$$f_2(x_1, x_2) = a_2 x_1^2 + (b_2 x_1 + e_2)x_2 + c_2 x_2^2 + d_2 x_1 + cst_2 \quad (3.b)$$

Applying the optimality condition, $\sum_{i=1}^{k} \omega_i \nabla f_i(\mathbf{X}^*) = 0$, to Eq. (3), results:

$$\begin{bmatrix} 2a_1 x_1 + b_1 x_2 + d_1 & 2a_2 x_1 + b_2 x_2 + d_2 \\ b_1 x_1 + 2c_1 x_2 + e_1 & b_2 x_1 + 2c_2 x_2 + e_2 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4)$$

As the system of equations (4) is homogeneous, the non-trivial solution, with $\boldsymbol{\omega} \neq \mathbf{0}$, requires a singularity, i.e., the determinant of the coefficient matrix must be null:

$$\begin{vmatrix} 2a_1x_1 + b_1x_2 + d_1 & 2a_2x_1 + b_2x_2 + d_2 \\ b_1x_1 + 2c_1x_2 + e_1 & b_2x_1 + 2c_2x_2 + e_2 \end{vmatrix} = 0 \tag{5}$$

which results in the following quadratic curve for (x_1, x_2):

$$\alpha x_1^2 + (\beta x_1 + \varepsilon)x_2 + \gamma x_2^2 + \delta x_1 + \tau = 0 \tag{6}$$

where:

$$\begin{aligned}
&\alpha = 2(a_1b_2 - a_2b_1) \\
&\beta = 4(a_1c_2 - a_2c_1) \\
&\gamma = 2(b_1c_2 - b_2c_1) \\
&\delta = 2(a_1e_2 - a_2e_1) + (d_1b_2 - d_2b_1) \\
&\varepsilon = 2(d_1c_2 - d_2c_1) + (b_1e_2 - b_2e_1) \\
&\tau = (d_1e_2 - d_2e_1)
\end{aligned}$$

Function gradients $\nabla f_1(\mathbf{X})$ and $\nabla f_2(\mathbf{X})$ are parallel on the curve defined by Eq. (6), but they have to be opposite, resulting positive weights in Eqs. (4). Being the system singular, to find a relation between the weights ω_1 and ω_2 we can use only one of the equations as the other is its linear combination. Using the first equation, this relation can be deduced as:

$$\frac{\omega_2}{\omega_1} = -\frac{2a_1x_1 + b_1x_2 + d_1}{2a_2x_1 + b_2x_2 + d_2} \tag{7}$$

which have positive values if and only if

$$(2a_1x_1 + b_1x_2 + d_1)(2a_2x_1 + b_2x_2 + d_2) < 0 \tag{8}$$

Therefore Eq. (6) provides the locus where the functions gradients are parallel and Eq. (8) defines the Pareto set for the two quadratic functions minimization problem.

The upper bound of Eq.(8)

$$(2a_1x_1 + b_1x_2 + d_1)(2a_2x_1 + b_2x_2 + d_2) = 0 \tag{9}$$

is reached if the first term $2a_1x_1 + b_1x_2 + d_1 = 0$ or the second $2a_2x_1 + b_2x_2 + d_2 = 0$. As both terms are the first components of ∇f_1 and ∇f_2, respectively, these conditions imply that the solution (x_1^*, x_2^*) is over $f_1(x_1, x_2)$ minimum or over $f_2(x_1, x_2)$ minimum.

Concluding, the Pareto set for quadratic functions will be a quadratic curve connecting the functions minima and where the gradients are parallel and opposite.

As an example, let us consider the following bi-objective problem:

minimize:

$$f_1(x_1, x_2) = 3x_1^2 + (x_1 + 1)x_2 + x_2^2 + 28x_1 + 69 \tag{10.a}$$

$$f_2(x_1, x_2) = x_1^2 - (x_1 + 1)x_2 + x_2^2 - 7x_1 + 19 \tag{10.b}$$

From Eq. 6, the Pareto set takes the form:

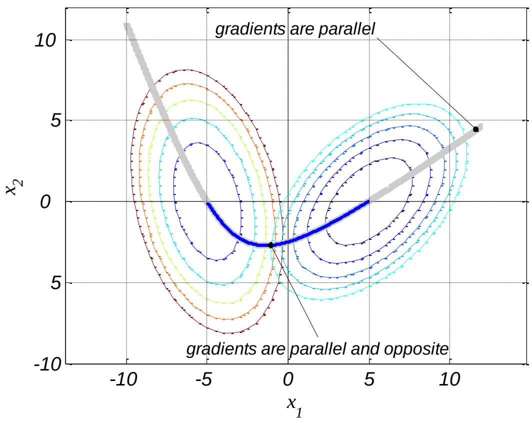
$$-8x_1^2 + (8x_1 + 70)x_2 + 4x_2^2 - 29x_1 - 21 = 0 \tag{11}$$

and is constrained by the following inequality:

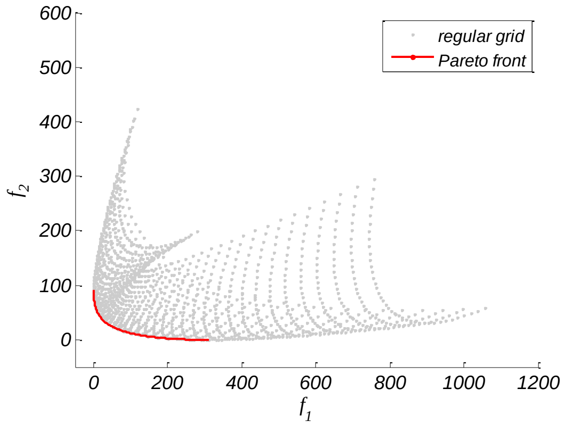
$$(6x_1 + x_2 + 28)(2x_1 - x_2 - 7) < 0 \qquad (12)$$

In Fig. 2.a is depicted the contours of functions f_1 and f_2 in the two-dimensional decision space. The thicker grey continuous curve represents Eq. (11) and the tick blue colour portion of this curve satisfies Eq. (12), being as expected the continuous Pareto set, namely, the curve along which the gradient vectors are parallel and opposite. In Figure 2.b, the continuous curve is the image of the Pareto set in the function space, i.e., the Pareto front. In addition, the blue dots are the functions image of a set of points taken in a regular grid in the design space.

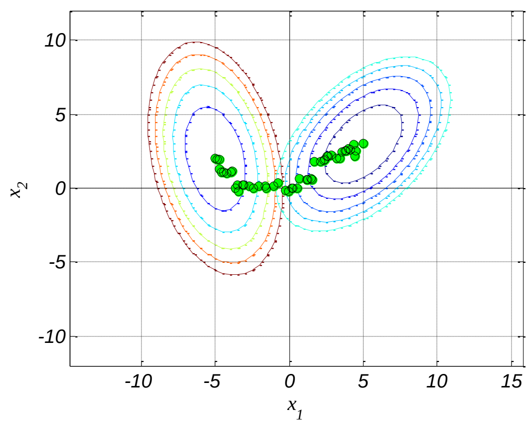
For comparison, it is shown in Figs 2.c e 2.d, adapted from Augusto et al (2012), the solution obtained by the genetic algorithm NSGA II Deb (2000). It can be seen that the points are evenly distributed in the function space but they are not in the decision space. That happens because the search procedure in most of GAs is focused in the function space, trying to get a well distributed Pareto front.



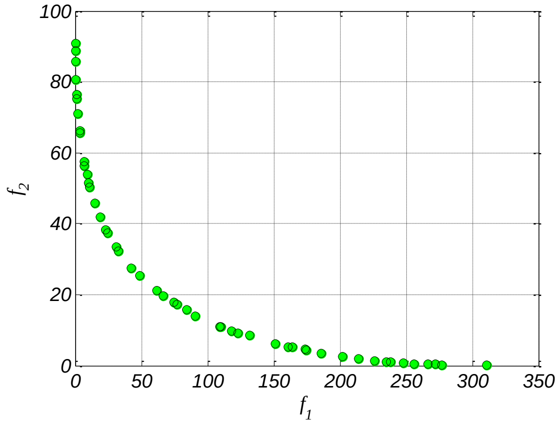
(a) - Continuous Pareto set obtained by the proposed method



(b) - Continuous Pareto front, the Pareto set image in the function space.



(c) - Pareto set for the performance functions f_1 and f_2 obtained by the NSGA II algorithm.



(d) - Pareto front for performance functions f_1 and f_2 obtained by the NSGA II algorithm.

Figure 1 - Graphical representation of *Proposition 1*. The continuous Pareto set as the locus where objective function gradients are parallel and opposite

3.2. Three or more functions defined in $\mathbb{R}^2$ space

In the previous section we found a closed form solution for the optimization of two quadratic functions in the bi-dimensional decision space. Unfortunately, we didn't find a similar solution when we add more functions in the problem. Nevertheless, the idea behind *Proposition 1* remains useful.

Consider a optimization problem involving three continuous differentiable functions f_1, f_2 and f_3. If a point $\mathbf{p}$ belongs to the Pareto set it must satisfy Eq. (2). Therefore, one gradient vector, $\nabla f_1(\mathbf{p})$, shall be a linear combination of the other two, $\nabla f_2(\mathbf{p})$ and $\nabla f_3(\mathbf{p})$, i.e., shall exist positive weights such that

$$\omega_1 \nabla f_1(\mathbf{p}) = -\omega_2 \nabla f_2(\mathbf{p}) - \omega_3 \nabla f_3(\mathbf{p}) \qquad (13)$$

In Fig. 3 it is illustrated such condition whit the gradient vectors $\nabla f_1(\mathbf{p})$, $\nabla f_2(\mathbf{p})$ and $\nabla f_3(\mathbf{p})$ associated with their weighted factors ω_1, ω_2 and ω_3, respectively.

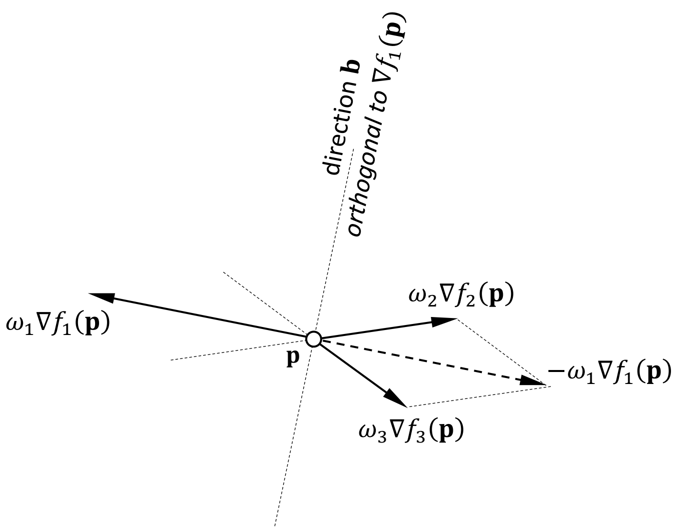


Figure 3 – Pareto optimality condition for three or more functions in $\mathbb{R}^2$ decision space.

An equilibrium condition exists when $\nabla f_1(\mathbf{p})$ is oriented through the opposite angular sector defined by the two other gradient vectors, namely, $\nabla f_2(\mathbf{p})$ and $\nabla f_3(\mathbf{p})$.

Based on this idea we suggest the following:

> ***Proposition 2***. *Let* $\mathbf{e}_i$ *be the unit vector defined by* $\mathbf{e}_i = \nabla f_i(\mathbf{p})/\|\nabla f_i(\mathbf{p})\|$, with $\|\nabla f_i(\mathbf{p})\| \neq \mathbf{0}$, *and* $\mathbf{e}_b$, *the unit vector orthogonal to* $\mathbf{e}_i$, *i.e.,* $\mathbf{e}_b \cdot \mathbf{e}_i = 0$. *If* $\mathbf{p}$ *belongs to the Pareto set resulting from the multiobjective optimization problem involving continuous and differentiable functions defined in* $\mathbb{R}^2$, *then exist at least three unit vectors, say* $\mathbf{e}_i(\mathbf{p}), \mathbf{e}_j(\mathbf{p})$ *and* $\mathbf{e}_l(\mathbf{p})$ *that satisfy the following conditions:*

$$(\mathbf{e}_j \cdot \mathbf{e}_i) < 0 \qquad (14.a)$$
$$(\mathbf{e}_l \cdot \mathbf{e}_i) < 0 \qquad (14.b)$$
$$(\mathbf{e}_j \cdot \mathbf{e}_b)(\mathbf{e}_l \cdot \mathbf{e}_b) < 0 \qquad (14.c)$$

The direction of $\mathbf{e}_b$ divides the decision space in two semi planes. If the vector $\nabla f_i(\mathbf{p})$ points to one side then Eqs. (14.a) e (14.b) state that the vectors $\nabla f_j(\mathbf{p})$ and $\nabla f_l(\mathbf{p})$ point to the other side and Eq. (14.c) states that -$\nabla f_i(\mathbf{p})$ is placed between them.

Equations (14) form a condition test for a point be Pareto optimum or not. This test can be useful if the problem has few optimization functions as to explore all distinguished set with three gradient vectors in a problem with k objective functions the maximum of $k!/(k-3)!$ permutations of i, j, l must be checked.

Let us apply *Proposition 2* to find the solution of an unconstrained MOOP with three quadratic objective functions being two of them those defined by Eqs. (10) and the third defined by:

$$f_3(x_1, x_2) = x_1^2 + 12x_2 + x_2^2 + 4x_1 + 40 \tag{15}$$

In the Fig. 4 is shown the Pareto set found applying the Pareto test in the points of a regular grid in the design space divided in fifty points in each coordinate axis, $(x_1, x_2) \in (-10,10]$ for all $(x_1, x_2)_{i,j} = \left(-10 + \frac{20}{50}i, -10 + \frac{20}{50}j\right)$, $i, j = 1..50$. The continuous border of the Pareto set were obtained applying *Proposition 1* for each pair of objective functions.

3.3. Quadratic functions defined in $\mathbb{R}^n$ space

In the former two sections we have considered unconstrained MOOP with quadratic functions defined in the two dimensional space. To proceed to larger dimensions, let us define a quadratic function in the $\mathbb{R}^n$ space, $f(\mathbf{X}): \mathbb{R}^n \to \mathbb{R}$, written as:

$$f(\mathbf{X}) = \tfrac{1}{2}\mathbf{X}_L^{\mathrm{T}}\mathbf{A}\mathbf{X}_L + cst \tag{16}$$

with

$$\mathbf{X}_L = \mathbf{T}(\mathbf{X} - \mathbf{X_0}) \tag{17}$$

and

$\mathbf{X}_L \in \mathbb{R}^n$ is a local coordinate system for a convenient definition of $f(\mathbf{X})$,

$\mathbf{X_0} \in \mathbb{R}^n$ is the position of the local coordinate system related to the global one,

$\mathbf{T}$ is the coordinates transformation matrix, from local to global coordinates systems.

Using Eq. (17), Eq. (16) can be rewritten:

$$f(\mathbf{X}) = \tfrac{1}{2}(\mathbf{X} - \mathbf{X_0})^{\mathrm{T}}\left(\mathbf{T}^{\mathrm{T}}\mathbf{A}\mathbf{T}\right)(\mathbf{X} - \mathbf{X_0}) + cst \tag{18}$$

Calling $\mathbf{A_r} = (\mathbf{T}^T\mathbf{A}\mathbf{T})$, Eq. (18) can be rewritten:

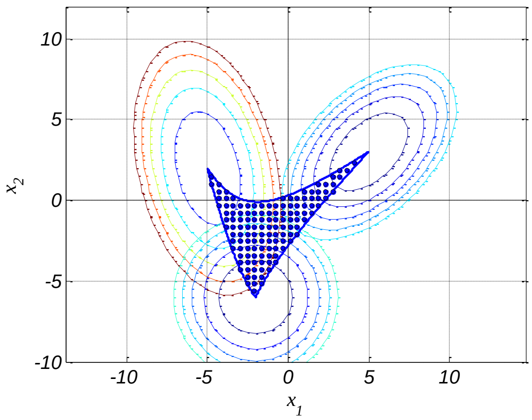
$$f(\mathbf{X}) = \tfrac{1}{2}(\mathbf{X} - \mathbf{X_0})^{\mathrm{T}}\mathbf{A_r}(\mathbf{X} - \mathbf{X_0}) + cst \tag{19}$$

As $f(\mathbf{X})$ is smooth, its gradient vector is:

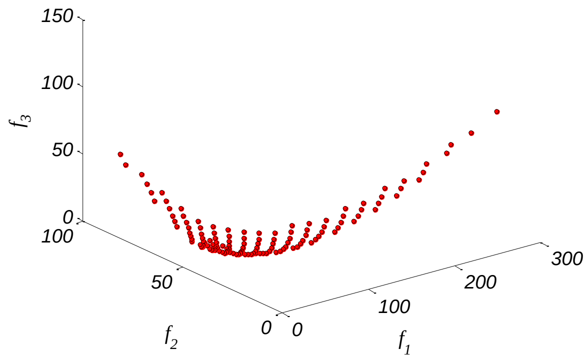
$$\nabla f(\mathbf{X}) = \mathbf{A_r}(\mathbf{X} - \mathbf{X_0}) \tag{20}$$

Matrix $\mathbf{A}$, as well as its transformed form $\mathbf{A_r}$, is the symmetric Hessian of $f(\mathbf{X})$, $\mathbf{H}(\mathbf{X})$, containing its second partial derivatives.

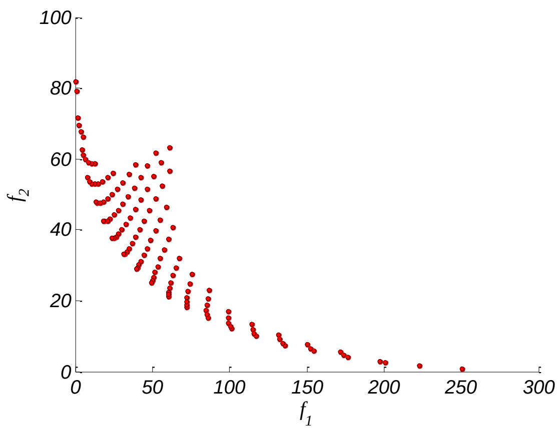
With these definitions, let $\mathbf{X}^*$ be the solution of an unconstrained MOOP involving k quadratic functions defined in $\mathbb{R}^n$ space. Accordingly, exists $\omega_i \geq 0$, $i = 1...k$, that satisfy the Eq. (2a), i.e.,



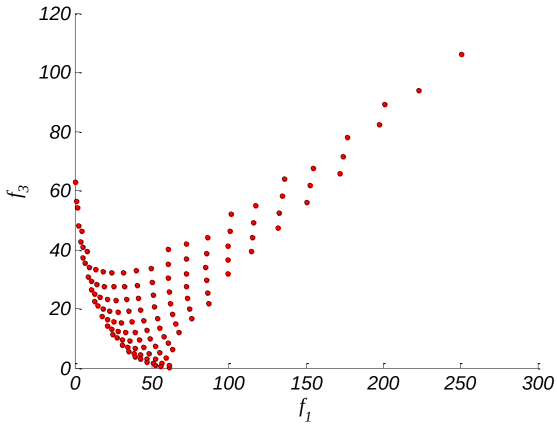
(a) - Pareto set



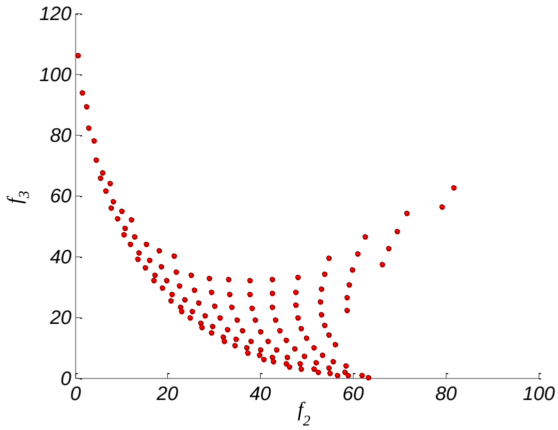
(b) - Pareto front



(c) Pareto front $f_1 - f_2$ view



(d) - Pareto front $f_1 - f_3$ view



(e) - Pareto front $f_2 - f_3$ view

Figure 4 – Pareto optimality condition applied to the three-objective optimization problem involving functions defined in the two-dimensional decision space.

$$\sum_{i=1}^{k} \omega_i \nabla f_i(\mathbf{X}^*) = 0 \tag{21}$$

As $f_i(\mathbf{X})$ is a quadratic, Eq. (20) can be used and Eq. (21) takes the form:

$$\sum_{i=1}^{k} \omega_i \mathbf{A}_{\mathbf{r}i}(\mathbf{X}^* - \mathbf{X}_{\mathbf{0}i}) = 0 \tag{22}$$

In Eq. (22), the weights ω_i as well as the searched solution $\mathbf{X}^*$ are unknown. Let us assume that all ω_i are known, i.e., $\omega_i = \omega_i^*$. Accordingly, Eq. (22) can be rewritten as:

$$\sum_{i=1}^{k} \omega_i^* \mathbf{A}_{\mathbf{r}i} \mathbf{X}^* = \sum_{i=1}^{k} \omega_i^* \mathbf{A}_{\mathbf{r}i}\, \mathbf{X}_{\mathbf{0}i} \tag{23}$$

Calling

$$\widehat{\mathbf{A}} = \sum_{i=1}^{k} \omega_i^* \mathbf{A}_{\mathbf{r}i} \tag{24}$$

and

$$\hat{\mathbf{b}} = \sum_{i=1}^{k} \omega_i^* \mathbf{A}_{\mathbf{r}i}\, \mathbf{X}_{\mathbf{0}i} \tag{25}$$

then, Eq.(23) can be rewritten as:

$$\widehat{\mathbf{A}}\mathbf{X}^* = \hat{\mathbf{b}} \tag{26}$$

Let us assume that all $\mathbf{A}_{\mathbf{r}i}$ are positive definite, i.e., $\mathbf{X}^T \mathbf{A}_{\mathbf{r}i} \mathbf{X} > 0$, for all $\mathbf{X} \in \mathbb{R}^n | \mathbf{X} \neq \mathbf{0}$. If ω_i^* is real, non-negative and satisfies the normalization equality, i.e., $\sum_{i=1}^{k} \omega_i^* = 1$, then $\widehat{\mathbf{A}}$ will also be positive definite and therefore its inverse $\widehat{\mathbf{A}}^{-1}$ will always exists.

Consequently, the Pareto optimum solution $\mathbf{X}^*$ can easily be found by solving Eq. (26), i.e.,

$$\mathbf{X}^* = \widehat{\mathbf{A}}^{-1}\hat{\mathbf{b}} \tag{27}$$

In this approach, we have considered that ω_i^* are known. Consequently, $\widehat{\mathbf{A}}$, Eq. (24), and $\hat{\mathbf{b}}$, Eq.(25), are promptly found. Although this is not the case for a general solution of Eq. (21), this approach is very useful to find the Pareto set and the Pareto front of unconstrained multiobjective optimization problems involving quadratic functions considering the following:

> ***Proposition 3***. *Consider a MOOP involving k quadratic functions, with the Hessian of each function being positive definite. To obtain n_p Pareto optimum solutions the following steps is proposed:*
>
> 1. *Sort, at random over the interval [0,1], the components of $\boldsymbol{\omega}^*$ a vector containing k weights ω_i^*.*
> 2. *Perform a normalization such as $\sum_{i=1}^{k} \omega_i^* = 1$.*
> 3. *Calculate $\widehat{\mathbf{A}} = \sum_{i=1}^{k} \omega_i^* \mathbf{A}_{\mathbf{r}i}$ and $\hat{\mathbf{b}} = \sum_{i=1}^{k} \omega_i^* \mathbf{A}_{\mathbf{r}i}\, \mathbf{X}_{\mathbf{0}i}$.*
> 4. *Solve the linear system $\mathbf{X}^* = \widehat{\mathbf{A}}^{-1}\hat{\mathbf{b}}$, getting the Pareto point $\mathbf{X}^*$ associated with $\boldsymbol{\omega}^*$.*
> 5. *Repeat, steps 1 to 4 for the number n_p of Pareto points wanted.*

Even requiring solutions of n_p linear systems, the method is very fast depending on the order of the matrix $\widehat{\mathbf{A}}$.

Before advance to the applications, consider an ellipsoid enclosed in a parallelepiped of size $2a$, $2b$ and $2c$, as shown in Fig. 5. Also consider a local coordinates system, $\mathbf{X}_L = [x_{L1}, x_{L2}, x_{L3}]^T$ with origin centered inside the ellipsoid, fixed to it and oriented along its semi-axes.

The family of quadratic functions that represents this ellipsoid can be written as

$$f(\mathbf{X}) = \tfrac{1}{2}\mathbf{X}_L^{\mathrm{T}}\mathbf{A}\mathbf{X}_L + cst = 0 \qquad (28)$$

with the matrix **A** defined in Fig. 5.

The ellipsoid can be rotated around the i^{th} coordinate axis, i.e., $\mathbf{X}_{\mathbf{r}_i} = \mathbf{r}_i\mathbf{X}_L$. Let α, β and θ be the rotation angles, around x_{L1}, x_{L2} and x_{L3} axes, respectively. Each individual rotation matrix is depicted in Figs. A1, A2 and A3, Annex A. Then, the general rotation matrix is defined by:

$$\mathbf{R} = \mathbf{r}_1(\alpha)\mathbf{r}_2(\beta)\mathbf{r}_3(\theta) \qquad (29)$$

$$\mathbf{A} = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{bmatrix}$$

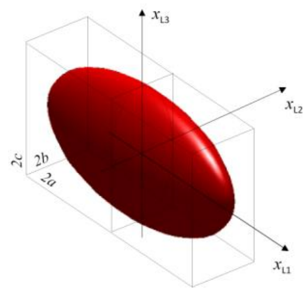


Figure 5 - Representation of an ellipsoid, a quadratic function $f(\mathbf{X}) = 0$ defined in $\mathbb{R}^3$ space.

The local coordinate system can be positioned at a point $\mathbf{X}_0$, relative to a global coordinates system, $\mathbf{X} = [x_1, x_2, x_3]^T$. In such a case, the points on the ellipsoid surface can be referred to the global coordinate system as

$$\mathbf{X} = \mathbf{R}\mathbf{X}_L + \mathbf{X}_0 \qquad (30)$$

To get the transformation matrix **T** of Eq. (17), we isolate $\mathbf{X}_L$ in Eq.(30), i.e.,

$$\mathbf{X}_L = \mathbf{R}^{-1}(\mathbf{X} - \mathbf{X}_0) \qquad (31)$$

Being **R** an orthogonal matrix, its inverse is equal to its transpose, i. e., $\mathbf{T} = \mathbf{R}^{-1} = \mathbf{R}^T$.

With the previous definitions, consider the following unconstrained MOOP:

$$\textit{minimize: } f_1(\mathbf{X}), f_2(\mathbf{X}) \textit{ and } f_3(\mathbf{X}) \qquad (32)$$

with $\mathbf{X} \in \mathbb{R}^3$, being $f_1(\mathbf{X}), f_2(\mathbf{X}), f_3(\mathbf{X})$ defined in Table 1 and illustrated in Fig. 6.a.

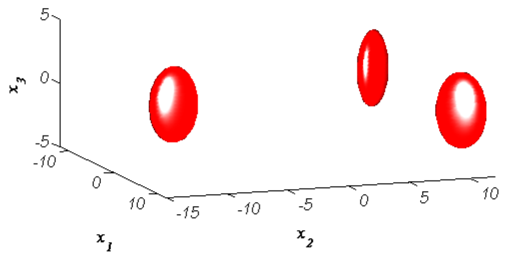
Table 1 – Coefficients for objective functions $f_i(\mathbf{X})$ definitions.

function	semi-axis			rotation			origin		
	a	b	c	α	β	θ	x_{01}	x_{02}	x_{03}
$f_1(\mathbf{X})$	1	2	3	0	0	-π/6	10	10	0
$f_2(\mathbf{X})$	1	2	3	0	0	0	0	-10	0
$f_3(\mathbf{X})$	1	2	3	0	0	π/4	-10	10	0

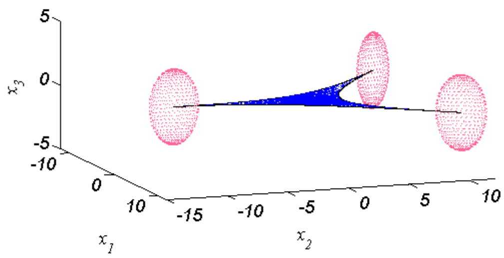
The Pareto set for this problem, illustrated in Fig. 6.b, were obtained applying *Proposition 3* algorithm, with n_p = 5000. To get all the points, an ordinary 2 GHz dual processor computer with 3 Gb RAM, running *Matlab* expended 0.99 seconds of processing time.

As all ellipsoids were placed over (x_1, x_2) plane and were rotated around x_3 axis, only, the Pareto set is over the (x_1, x_2) plane. Bold points at the Pareto set boundary were found with the same method applied to the functions $f_1(\mathbf{X}), f_2(\mathbf{X}), f_3(\mathbf{X})$ taken in pairs. According to *Proposition 1*, in such cases, the Pareto set is necessarily a curve.

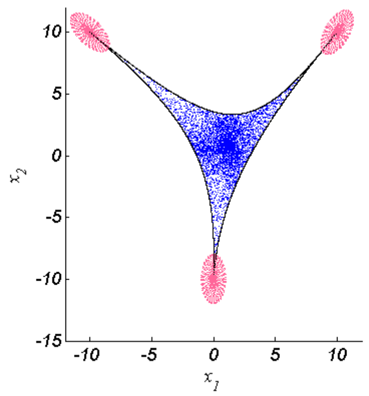
The Pareto front is shown in Fig. 6.d. It should be noticed that this front was obtained by means of a straightforward solution of the Pareto optimality conditions without using any iterative algorithm.



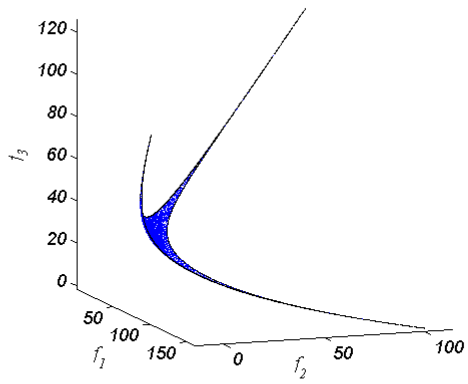
(a) –Three quadratic functions, $f_i(\mathbf{X}) = 0$, cst = - 0.5



(b) – Pareto set



(c) – Pareto set, $x_1 - x_3$ view.

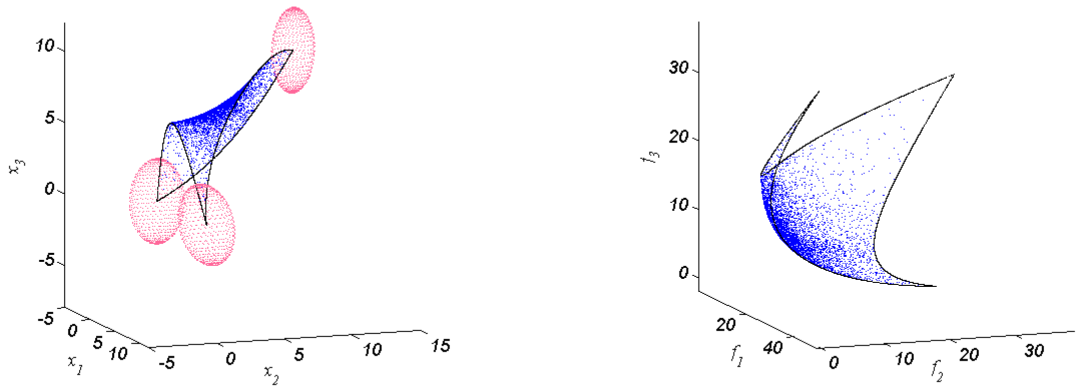


(d) – Pareto front.

Figure 6 – Solution of the unconstrained MOOP with the quadratic functions defined in Table 1

In the next example three ellipsoids with different orientations, as defined in Table A1, Annex A, were distributed in the (x_1, x_2, x_3) space.

The Pareto set of this optimization problem found by the proposed methodology delineates the curved surface shown in Fig. 7.a. The Pareto front, in the function space, is shown in Fig. 7.b.



(a) – Pareto set *(b) – Pareto front*

Figure 7 – Solution of the unconstrained MOOP with the quadratic functions defined in Table A1, Annex A

Adding to the unconstrained MOOP the function $f_4(\mathbf{X})$, defined in Table A2, *Annex* A, the proposed method generated in 1.17 seconds the three dimensional Pareto set illustrated in Fig. 8.

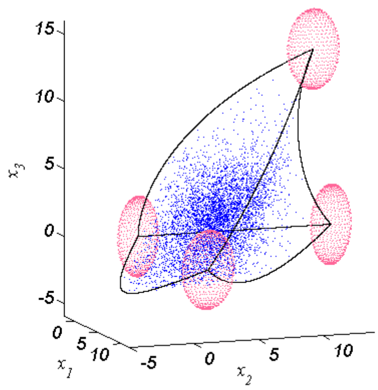


Figure 8 – Pareto set of the unconstrained MOOP with the quadratic functions defined in Table A2, Annex A

In the problems all functions were defined by convenience in $\mathbb{R}^3$ space, nevertheless, *Proposition 3* can be applied to quadratic functions defined in $\mathbb{R}^n$ space.

4. Conclusions

Most of the real problems are multiobjective and their objective functions being antagonistic. To solve this problem many researchers are developing methods to solve multiobjective optimization problems without reducing them to single objective. Up to now, evolutionary algorithms are widespread as a general technique to find a candidate set of the optimal solutions. These algorithms provide a discrete picture of the Pareto front in the function space, without bringing to much information about the decision space.

In the framework of this paper, we have proposed different methods to determine the Pareto set of unconstrained multiobjective optimization problems involving quadratic objective functions. Three different procedures were proposed. One for bi-objective optimization, with functions defined in $\mathbb{R}^2$ space, which results in an analytical solution for the Pareto set. For three or more functions also defined in $\mathbb{R}^2$ space it was proposed a condition test that is able to check if a point in the decision space is Pareto optimum or not. In the third method, suitable for multiobjective optimization with functions defined in $\mathbb{R}^n$ space and having Hessian positive definite, a direct algorithm was proposed which finds a Pareto optimum based in an arbitrary valid weighting vector. Some illustrative examples were used to highlight the potentiality of the methods.

It is apparent that the Pareto set for two distinct two-dimensional functions is a curve, and for three and above, the Pareto set is a surface. In three-dimensional space, for two distinct three-dimensional functions, the Pareto set will be a space curve, for three functions, a surface, and for four functions and above, a solid.

Although the proposed methods are restricted to unconstrained optimization the authors believe they can be extend to constrained problems and are working on it.

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Annex – A

$$\mathbf{r}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

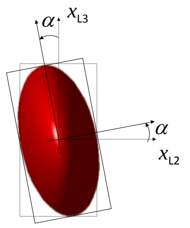

Figure A1 - Rotation α around x_{L1} axis.

$$\mathbf{r}_2(\beta) = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

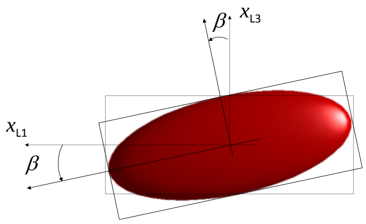

Figure A2 - Rotation β around x_{L2} axis.

$$\mathbf{r}_3(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

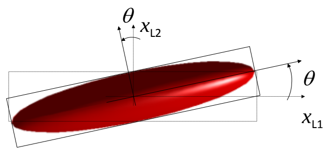

Figure A3 - Rotation θ around x_{L3} axis.

Table A1 – Optimization problem with 3 objective functions

function	semi-axis			rotation			origin		
	a	b	c	α	β	θ	x_{01}	x_{02}	x_{03}
$f_1(\mathbf{X})$	1	2	3	0	0	0	0	0	0
$f_2(\mathbf{X})$	1	2	3	0	π/4	0	10	0	0
$f_3(\mathbf{X})$	1	2	3	0	0	π/6	0	10	10

Table A2 – Optimization problem with 4 objective functions

function	semi-axis			rotation			origin		
	a	b	c	α	β	θ	x_{01}	x_{02}	x_{03}
$f_1(\mathbf{X})$	1	2	3	0	0	π/6	0	0	0
$f_2(\mathbf{X})$	1	2	3	0	-π/30	0	15	0	0
$f_3(\mathbf{X})$	1	2	3	0	0	π/6	0	15	0
$f_4(\mathbf{X})$	1	2	3	0	0	0	10	10	15